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Predual of Campanato spaces and Riesz potentials

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1. Introduction

This is an announcement of my recent work.

Let $X=(X,\delta,\mu)$ be a space of homogeneous type (SHT), i.e. X is a topological space endowed with a quasi-distance δ and a nonnegative measure μ such that

$$\delta(x,y)\ge 0 \quad \text{and} \quad \delta(x,y)=0 \text{ if and only if } x=y,$$

$$\delta(x,y)=\delta(y,x),$$

$$\delta(x,y)\le K_1\,(\delta(x,z)+\delta(z,y)), \tag{1.1}$$

the balls $B(x,r)=\{y\in X:\delta(x,y)<r\}$, $r>0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains all balls, and

$$0<\mu(B(x,2r))\le K_2\,\mu(B(x,r))<\infty, \tag{1.2}$$

where $K_i\ge 1$ $(i=1,2)$ are constants independent of $x,y,z\in X$ and $r>0$.

If there are constants θ $(0<\theta\le 1)$ and $K_3\ge 1$ such that

$$|\delta(x,z)-\delta(y,z)|\le K_3\,(\delta(x,z)+\delta(y,z))^{1-\theta}\delta(x,y)^{\theta},\quad x,y,z\in X, \tag{1.3}$$

then the balls are open sets. The number θ is called the order of the SHT.

We shall say that a SHT is normal if there are constants $K_4>0$ and $K_5>0$

$$K_4 r\le \mu(B(x,r))\le K_5 r \quad \text{for } x\in X \text{ and } \mu(\{x\})<r<\mu(X). \tag{1.4}$$

We note that, for any SHT (X,d,μ), there exists a quasi-distance δ such that (X,δ,μ) is normal and of some order θ, and that the topologies induced on X by d and δ coincide (Macías and Segovia (1979)).

Let $X=\mathbb{R}^n$, $d(x,y)=|x-y|$ and μ be the Lebesgue measure. If $\delta(x,y)=|x-y|^n$, then $(\mathbb{R}^n,\delta,\mu)$ is normal and of order $1/n$.

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In this talk we always assume that (X, δ, μ) is normal and of order θ and that $\mu(\{x\}) = 0$ for all x in X.

We consider Riesz potentials

$$I_\alpha f(x) = \int_X \frac{f(y)}{\delta(x,y)^{1-\alpha}}\, d\mu(y),$$

for $0 < \alpha < \theta$. It is kown that the operator I_α is bounded from $L^p(X)$ to $L^q(X)$ if $1 < p < q < \infty$ and $-1/p + \alpha = -1/q$ (Gatto and Vagi(1990)). This boundedness is well known as the Hardy-Littlewood-Sobolev theorem in $\mathbb{R}^n$ case.

In this report, we define a generalized Hardy space $H_U^{[\phi,q]}(X)$ and investigate continuity of I_α on $H_U^{[\phi,q]}(X)$. We show

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathcal{L}_{q',\phi}(X),$$

where $\mathcal{L}_{q',\phi}(X)$ is a Campanato space. Campanato spaces are Banach spaces modulo constants, which include $\mathrm{BMO}(X)$ and $\mathrm{Lip}_\alpha(X)$ as special cases.

We first define I_α for functions $f \in \mathcal{L}_{q',\phi}(X)$. To do this we define the modified version of I_α as follows;

$$\tilde{I}_\alpha f(x) = \int_X f(y) \left(\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1-\chi_{B_0}(y)}{\delta(x_0,y)^{1-\alpha}} \right) dy,$$

where $B_0 = B(x_0, r_0)$ is a fixed ball. We can show that $\tilde{I}_\alpha f(x)$ converges absolutely for all x and therefore changing B_0 in the definition above results in adding a constant. We assume that δ satisfies the cancellation property;

$$\int_X \left(\frac{1}{\delta(x,y)^{1-\alpha}} - \frac{1}{\delta(x',y)^{1-\alpha}} \right) d\mu(y) = 0 \quad \text{for any } x, x' \text{ in } X. \tag{1.5}$$

In case of $X = \mathbb{R}^n$ or $\mathbb{T}^n$, (1.5) holds for $\delta(x,y) = |x-y|^n$ and for $0 < \alpha < 1$. For other examples of spaces of homogeneous type with the property (1.5), see [3]. We note that, for all normal spaces (X, δ, μ) with $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$, we can fined a quasi-distance δ_α equivalent to δ, such that (1.5) holds (see [2]).

2. Campanato spaces $\mathcal{L}_{p,\phi}(X)$ and Hölder spaces $\Lambda_\phi(X)$

Let $1 \le p < \infty$ and $\phi : X \times (0,\infty) \to (0,\infty)$. For a ball $B = B(x,r)$, we shall write $\phi(B)$ in place of $\phi(x,r)$. The function spaces $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y \in X,\, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, \delta(x,y)) + \phi(y, \delta(y,x))},$$

and

$$f_B = \mu(B)^{-1} \int_B f(x) \, d\mu(x).$$

Then $\mathcal{L}_{p,\phi}(X)$ and $\Lambda_\phi(X)$ are Banach spaces modulo constants with the norms $\|f\|_{\mathcal{L}_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively. If $p = 1$ and $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}(X)$.

Let $\mathcal{G}_*$ be the set of all functions $\phi : X \times (0, \infty) \to (0, \infty)$ such that

$$\frac{1}{A_1} \leq \frac{\phi(x,s)}{\phi(x,r)} \leq A_1, \qquad \frac{1}{2} \leq \frac{s}{r} \leq 2, \tag{2.1}$$

$$\phi(x,r) \leq A_2 \phi(y,s), \qquad B(x,r) \subset B(y,s), \tag{2.2}$$

where A_1 and $A_2 > 0$ are independent of $r, s > 0$, $x, y \in X$.

Theorem 2.1. *Let $\phi \in \mathcal{G}_*$. Then*

$$\mathcal{L}_{p,\phi}(X) = \mathcal{L}_{1,\phi}(X)$$

with equivalent norms for every $1 \leq p < \infty$.

Theorem 2.2. *Let $\phi \in \mathcal{G}_*$ and there exists $C > 0$ such that*

$$\int_0^{\delta(x,y)} \frac{\phi(x,t)}{t} \, dt \leq C \phi(x, \delta(x,y)), \quad x, y \in X. \tag{2.3}$$

Then

$$\Lambda_\phi(X) = \mathcal{L}_{p,\phi}(X)$$

with equivalent norms for every $1 \leq p < \infty$.

We say that $\alpha(\cdot) : X \to [0, \infty)$ is log-Hölder continuous if there exists $C_0 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_0}{\log(1/\delta(x,y))} \quad \text{for} \quad \delta(x,y) < 1/2. \tag{2.4}$$

Let $\alpha_- = \inf_{x \in X} \alpha(x)$ and $\alpha_+ = \sup_{x \in X} \alpha(x)$.

Example 2.1. Let $\alpha(\cdot)$ be log-Hölder continuous and

$$\phi(x,r) = r^{\alpha(x)} \quad \text{with} \quad 0 < \alpha_- \leq \alpha_+ \leq \theta.$$

Then $\phi \in \mathcal{G}_*$ and satisfies (2.3). In this case we denote $\Lambda_\phi(X)$ by $\mathrm{Lip}_{\alpha(\cdot)}(X)$ and

$$\|f\|_{\mathrm{Lip}_{\alpha(\cdot)}} = \sup_{x,y\in X,\, x\neq y} \frac{2|f(x)-f(y)|}{\delta(x,y)^{\alpha(x)} + \delta(y,x)^{\alpha(y)}}.$$

If $\alpha(x) \equiv \alpha$, then $\mathrm{Lip}_{\alpha(\cdot)}(X) = \mathrm{Lip}_\alpha(X)$.

3. Generalized Hardy spaces $H_U^{[\phi,q]}(X)$

Let $\phi : X \times (0,\infty) \to (0,\infty)$, $1 < q \le \infty$ and $1/q + 1/q' = 1$.

Definition 3.1 ($[\phi,q]$-atom). A function a on X is called a $[\phi,q]$-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset B$,

(ii) $\|a\|_q \le \dfrac{1}{\mu(B)^{1/q'}\phi(B)}$,

(iii) $\displaystyle\int_X a(x)\, d\mu(x) = 0$,

where $\|a\|_q$ is the L^q norm of a. We denote by $A[\phi,q]$ the set of all $[\phi,q]$-atoms.

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0,\infty) \to [0,\infty)$. Then $\Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \infty$ for all $\Phi \in \mathcal{F}$.

Let $\mathcal{F}_X$ be the set of all functions $\Phi : X \times [0,\infty) \to [0,\infty)$ such that

(i) $\Phi(x,\cdot) \in \mathcal{F}$ for every $x \in X$, and

(ii) $\Phi(\cdot, r)$ is measurable on X for all $r \in [0,\infty)$.

We denote by $\Phi^{-1}(x,\cdot)$ the inverse of $\Phi(x,\cdot)$ with respect to $r \in [0,\infty)$.

For $\Phi \in \mathcal{F}_X$ and $B = B(x,r)$, let

$$\phi(x,r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(x, 1/\mu(B))}. \tag{3.1}$$

Then

$$\frac{1}{\mu(B)^{1/q'}\phi(B)} = \mu(B)^{1/q}\Phi^{-1}\left(x, \frac{1}{\mu(B)}\right).$$

If $\Phi(x,r) = r^{p(x)}$, $p(\cdot) : X \to (0,1]$, then

$$\frac{1}{\mu(B)^{1/q'}\phi(B)} = \mu(B)^{1/q - 1/p(x)}.$$

If $\Phi(x,r) = r^p$, $0 < p \le 1$, then

$$\frac{1}{\mu(B)^{1/q'}\phi(B)} = \mu(B)^{1/q-1/p}.$$

In this case, $[\phi,q]$-atoms are the usual (p,q)-atoms.

We define $H_U^{[\phi,q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q',\phi}(X)$. We can see $A[\phi,q] \subset (\mathcal{L}_{q',\phi}(X))^*$ as follows. If a is a $[\phi,q]$-atom and a ball B satisfies (i)–(iii), then

$$
\begin{aligned}
(3.2)\qquad \left|\int_X a(x)g(x)\,d\mu(x)\right| &= \left|\int_B a(x)(g(x)-g_B)\,d\mu(x)\right| \\
&\le \|a\|_q \left(\int_B |g(x)-g_B|^{q'}\,d\mu(x)\right)^{1/q'} \\
&\le \frac{1}{\phi(B)}\left(\frac{1}{\mu(B)}\int_B |g(x)-g_B|^{q'}\,d\mu(x)\right)^{1/q'} \\
&\le \|g\|_{\mathcal{L}_{q',\phi}}.
\end{aligned}
$$

That is, the mapping $g \mapsto \int_X ag\,d\mu$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(X)$ with norm not exceeding 1.

Definition 3.2 ($H_U^{[\phi,q]}(X)$). Let $\phi : X\times(0,\infty) \to (0,\infty)$, $1 < q \le \infty$, $1/q+1/q' = 1$ and $U \in \mathcal{F}$ be concave. We define the space $H_U^{[\phi,q]}(X) \subset (\mathcal{L}_{q',\phi}(X))^*$ as follows:

$f \in H_U^{[\phi,q]}(X)$ if and only if there exist sequences $\{a_j\} \subset A[\phi,q]$ and positive numbers $\{\lambda_j\}$ such that

$$
(3.3)\qquad f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(X))^* \text{ and } \sum_j U(\lambda_j) < \infty.
$$

From $U(0) = 0$ and the concavity of U it follows that

$$
(3.4)\qquad U(Cr) \le CU(r), \quad 1 \le C < \infty,\ 0 \le r < \infty,
$$

$$
(3.5)\qquad U(r+s) \le U(r) + U(s), \quad 0 \le r,s < \infty.
$$

Then $H_U^{[\phi,q]}(X)$ is a linear space.

In general, the expression (3.3) is not unique. We define

$$
\|f\|_{H_U^{[\phi,q]}} = \inf\left\{U^{-1}\left(\sum_j U(\lambda_j)\right)\right\},
$$

where the infimum is taken over all expressions as in (3.3). We note that $\|f\|_{H_U^{[\phi,q]}}$ is not a norm in general. Let $m(f,g) = U(\|f-g\|_{H_U^{[\phi,q]}})$ for $f,g \in H_U^{[\phi,q]}(X)$. Then $m(f,g)$ is a metric and $H_U^{[\phi,q]}(X)$ is complete with respect to this metric.

If $\phi(B) = \mu(B)^{1/p-1}$ and $U(r) = r^p$, then $H_U^{[\phi,q]}(X)$ coinsides $H^{p,q}(X)$ defined by Coifman and Weiss (1977). They showed $H^{p,q}(X) = H^{p,\infty}(X)$ with equivalent metrics when $0 < p \le 1 < q \le \infty$ and denoted this space by $H^p(X)$. We extend this result to $H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X)$ in the next section.

Let $I(r) = r$. Then $\|f\|_{H_I^{[\phi,q]}}$ is a norm and $H_I^{[\phi,q]}(X)$ is a Banach space, which was defined by Zorko (1986) in the case $X = \mathbb{R}^n$. Therefore, our definition is a generalization of both definitions.

From the definition we have the following relations.

Proposition 3.1. (i) *If $1 < q_1 < q_2 \leq \infty$, then*

$$H_U^{[\phi,q_2]}(X) \subset H_U^{[\phi,q_1]}(X).$$

(ii) *If $\psi(B) \leq C\phi(B)$ for all balls B, then*

$$H_U^{[\phi,q]}(X) \subset H_U^{[\psi,q]}(X).$$

(iii) *If $V(r) \leq CU(r)$ for $0 \leq r \leq 1$, then*

$$H_U^{[\phi,q]}(X) \subset H_V^{[\phi,q]}(X).$$

(iv) *For any concave function $U \in \mathcal{F}$,*

$$H_U^{[\phi,q]}(X) \subset H_I^{[\phi,q]}(X).$$

In the above, the inclusion mapping are continuous.

4. Equivalence $H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X)$

Theorem 4.1. *Let $\phi \in \mathcal{G}_*$. If there exists $C_* > 0$ such that*

$$U(rs) \leq C_* U(r)U(s) \quad \text{for} \quad 0 < r, s \leq 1, \tag{4.1}$$

$$U\left(\frac{\mu(B_1)\phi(B_1)}{\mu(B_2)\phi(B_2)}\right) \leq C_* \frac{\mu(B_1)}{\mu(B_2)} \quad \text{for } B_1 \subset B_2, \tag{4.2}$$

then

$$H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X),$$

with equivalent topologies.

For $\Phi(x, r) \in \mathcal{F}_X$, let

$$\phi(x, r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(x, 1/\mu(B))}.$$

Example 4.1. Assume that $\mu(X) < \infty$. Let $p(\cdot)$ be log-Hölder continuous and

$$\Phi(x, r) = r^{p(x)}, \quad U(r) = r^{p_+} \quad \text{with} \quad 0 < p_- \leq p_+ \leq 1.$$

Then the assumption of Theorem 4.1 holds. Therefore

$$H_U^{[\phi,q]}(X) = H_U^{[\phi,\infty]}(X).$$

In this case we denote $H_U^{[\phi,q]}(X)$ by $H^{p(\cdot)}(X)$. If $p(\cdot) \equiv p$, then $H^{p(\cdot)}(X) = H^p(X)$, the usual Hardy space.

5. Duality

Let $L_c^q(X)$ be the set of all L^q-functions with bounded support, and let

$$L_c^{q,0}(X) = \left\{ f \in L_c^q(X) : \int_X f \, d\mu = 0 \right\}.$$

Then, for $1 < q \leq \infty$, $L_c^{q,0}(X)$ is dense in $H_U^{[\phi,q]}(X)$.

If $g \in \mathcal{L}_{q',\phi}(X)$ and $f \in L_c^{q,0}(X)$, then $f(g+c)$ is integrable for all constants c and $\int_X f(g+c)\, d\mu$ is independent of c.

Theorem 5.1. *If U satisfies*

$$\sup_{0<s\leq 1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0), \tag{5.1}$$

then

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathcal{L}_{q',\phi}(X).$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)$, then the mapping $\ell : f \mapsto \int_X f(g+c)\, d\mu$, for $f \in L_c^{q,0}(X)$, can be extended to a continuous linear functional on $H_U^{[\phi,q]}(X)$. Conversely, if ℓ is a continuous linear functional on $H_U^{[\phi,q]}(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)$ such that $\ell(f) = \int_X f(g+c)\, d\mu$ for $f \in L_c^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 5.2. *Let $\phi \in \mathcal{G}_*$. Then, for any $q \in (1,\infty]$ and for any concave function $U \in \mathcal{F}$ with* (5.1),

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathcal{L}_{1,\phi}(X).$$

Corollary 5.3. *Let $\phi \equiv 1$. Then, for any $q \in (1,\infty]$ and for any concave function $U \in \mathcal{F}$ with* (5.1),

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathrm{BMO}(X).$$

Corollary 5.4. *Let $\phi \in \mathcal{G}_*$ and there exists $C > 0$ such that*

$$\int_0^{\delta(x,y)} \frac{\phi(x,t)}{t}\, dt \leq C\phi(x,\delta(x,y)), \quad x,y \in X.$$

Then, for any $q \in (1,\infty]$ and for any concave function $U \in \mathcal{F}$ with (5.1),

$$\left(H_U^{[\phi,q]}(X)\right)^* = \Lambda_\phi(X).$$

Example 5.1. Under the assumption of Example 4.1, let $\alpha(x) = 1/p(x) - 1$. Then

$$\left(H^{p(\cdot)}(X)\right)^* = \mathrm{Lip}_{\alpha(\cdot)}(X).$$

6. Equivalence $H_U^{[\phi,q]}(X,d,\mu) = H_U^{[\psi,q]}(X,\delta,\mu)$

For a space of homogeneous type (X, d, μ) such that the balls are open sets, let

$$\delta(x,y) = \begin{cases} \inf\{\mu(B^d) : \ B^d \ni x, y \ \} & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases} \tag{6.1}$$

where B^d denotes a ball by the quasi-distance d. Then (X, δ, μ) is normal and the topologies induced on X by d and δ coincide.

Theorem 6.1. *Suppose that $\psi : X \times (0,\infty) \to (0,\infty)$ satisfies (2.1). Let $\tilde{\phi}(x,r) = \phi(x, \mu(B^d(x,r)))$. Then*

$$\mathcal{L}_{p,\tilde{\phi}}(X,d,\mu) = \mathcal{L}_{p,\phi}(X,\delta,\mu),$$
$$H_U^{[\tilde{\phi},q]}(X,d,\mu) = H_U^{[\phi,q]}(X,\delta,\mu),$$

with equivalent topologies, respectively.

Example 6.1. Let $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and μ be the Lebesgue measure. Then

$$\delta(x,y) = \frac{v_n}{2^n}|x-y|^n,$$
$$\tilde{\phi}(x,r) = \phi(x, v_n r^n),$$

where v_n is the volume of the unit ball. Therefore, $(\mathbb{R}^n, \delta, \mu)$ is of order $1/n$ and, for $0 < \alpha < \theta = 1/n$,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\delta(x,y)^{1-\alpha}}\, d\mu(y) = \int_{\mathbb{R}^n} \frac{f(y)}{(\frac{v_n}{2^n}|x-y|^n)^{1-\alpha}}\, d\mu(y).$$

7. Riesz potentials on $\mathcal{L}_{p,\phi}(X)$

Theorem 7.1. *Let $0 < \alpha < \theta$, $1 \le p < \infty$ and $\phi, \psi \in \mathcal{G}_*$. Assume that there exists a constant $A > 0$ such that, for all $x \in X$ and $r > 0$,*

$$r^\theta \int_r^\infty \frac{t^\alpha \phi(x,t)}{t^{1+\theta}}\, dt \le A\psi(x,r). \tag{7.1}$$

Then $\tilde{I}_\alpha$ is bounded from $\mathcal{L}_{p,\phi}(X)$ to $\mathcal{L}_{p,\psi}(X)$.

Corollary 7.2. *Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $\beta(\cdot)$ and $\gamma(\cdot)$ are log-Hölder continuous and*

$$\alpha + \beta(x) = \gamma(x) \quad \textit{with} \quad 0 < \beta_- < \gamma_+ < \theta.$$

Then $\tilde{I}_\alpha$ is bounded from $\mathrm{Lip}_{\beta(\cdot)}(X)$ to $\mathrm{Lip}_{\gamma(\cdot)}(X)$.

8. Riesz potentials on $H_U^{[\phi,\infty]}(X)$

Theorem 8.1. *Let $0 < \alpha < \theta$, $\phi, \psi \in \mathcal{G}_*$ and $U, V \in \mathcal{F}$ be concave. Assume that there exist $0 < \epsilon < 1$, $0 < \tau \le 1$ and $A > 0$ such that*

$$\psi(x,r)r^\alpha \le A\phi(x,r), \quad r > 0, \tag{8.1}$$

$$s^{\alpha-\theta-1}\left(s\psi(x,s)\right)^{1/\epsilon} \le Ar^{\alpha-\theta-1}\left(r\psi(x,r)\right)^{1/\epsilon}, \quad 0 < r \le s, \tag{8.2}$$

$$V(r) \le Ar^\tau, \quad r \in (0,1], \tag{8.3}$$

$$V(rs) \le AV(r)U(s), \quad 0 \le r, s \le 1. \tag{8.4}$$

Then there exists $C > 0$ such that

$$\|I_\alpha a\|_{H_V^{[\psi,\infty]}} \le C \quad \textit{for all } a \in A[\phi,\infty],$$

and I_α extends to a continuous linear map from $H_U^{[\phi,\infty]}(X)$ to $H_V^{[\psi,\infty]}(X)$.

Corollary 8.2. *Let $\mu(X) < \infty$, $0 < \alpha < \theta$. Assume that $p(\cdot)$ and $q(\cdot)$ are log-Hölder continuous and*

$$-\frac{1}{p(x)} + \alpha = -\frac{1}{q(x)} \quad \textit{with} \quad \frac{1}{1+\theta} < p_- < q_+ \le 1. \tag{8.5}$$

Then there exists $C > 0$ such that

$$\|I_\alpha a\|_{H^{q(\cdot)}} \le C \quad \textit{for all } a \in A(p(\cdot),\infty),$$

and I_α extends to a continuous linear map from $H^{p(\cdot)}(X)$ to $H^{q(\cdot)}(X)$.

In the above, $a \in A(p(\cdot),\infty)$ means that there exists $B = B(x,r)$ such that

(i) $\operatorname{supp} a \subset B$,
(ii) $\|a\|_q \le \mu(B)^{1/q-1/p(x)}$,
(iii) $\displaystyle\int_X a(x)\,d\mu(x) = 0$.

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